Using Mode Matching Methods in Horn Loudspeaker Simulation

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Summary
A method for simulating an expanding duct like a horn by mode matching techniques is described. The horn is described by a series of cylindrical or rectangular straight ducts, making a staircase approximation to the horn profile. The sound field in each pipe can be described as mode sums. At each discontinuity, the modes are coupled, and the horn is terminated by a modal radiation impedance.

This paper describes how the horn throat impedance and radiated pressure from a rectangular horn loudspeaker can be simulated using this method. Two cases are investigated: the horn mounted in an infinite baffle, and mounted in a small baffle or flange. For the latter case, edge diffraction from the baffle edges is taken into account for both the radiation impedance and the radiated pressure. Measurement results are presented, and are shown to be in good agreement with the simulations for both cases.

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1. Introduction
As has been pointed out by several authors [1, 2], the classical, one-dimensional horn theory, based on what is known as Webster’s horn equation\(^1\), is not able to predict the sound field radiated by a general horn.

By instead using the full wave equation, and letting the horn walls follow the coordinate surfaces of a coordinate system where the wave equation is separable, analytical solutions, including higher order modes of propagation, can be found. There are, however, only eleven coordinate systems where the wave equation is separable, and very few of them have surfaces that give useful horn contours [1].

A solution to this problem is to divide the horn into small sections that each have simple eigenmode functions, and then to couple the higher order modes between the sections. This approach was probably first implemented by Alfredson [4], who used an iterative technique. Another method, which is similar to the one described by Pagneux et al. [5], is described by Kemp [6]. The method has mainly been used for horn musical instrument simulations, using a cylindrical geometry, but a few researchers have also used it for loudspeaker simulations, see Shindo et. al [7] and Schuhmacher and Rasmussen [8]. These researchers used a rectangular geometry, and the method is implemented in a quite different way from Pagneux and Kemp.

In this paper, the mode-matching method described by Pagneux and Kemp will be verified by measurements, and it will be shown how edge diffraction can be included to account for finite baffles. It will be shown that this method can give an accurate prediction of the throat impedance and radiated sound field of a rectangular horn.

2. Theory
The Mode Matching Method (MMM) is based on describing the sound field in connected duct sections of different cross sections by a weighted sum of allowed modes [6, 9, 10]. In a rectangular duct, there will be \(n_x\) by \(n_y\) nodal lines, and \((n_x, n_y) = (0, 0)\) represents the plane wave mode.

Along the duct, assuming propagation in the \(z\) direction, the pressure can be expressed as a sum of all the modes:

\[
p(x, y, z) = \sum_{n=0}^{\infty} P_n(z) \psi_n(x, y)
\]

where \(P_n\) is the pressure profile along the tube, and \(\psi_n\) is the pressure profile in the \((x, y)\) plane. It is advantageous to separate \(\psi_n\) into two parts, one dependent on \(x\), and the other on \(y\):

\[
\psi_n = \phi_{n_x} \sigma_{n_y}
\]

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\(^1\) Although the equation in question was derived and discussed by Bernoulli, Lagrange and Euler [3].

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In symmetric ducts having hard wall boundary conditions, half-width $a$ and half-height $b$, the wave functions in (2) are [10]

$$\phi_{n_x} = \begin{cases} 
1 \sqrt{2} \cos \left( \frac{n_x \pi x}{a} \right) : & n_x = 0 \\
\sqrt{2} \cos \left( \frac{n_x \pi y}{a} \right) : & n_x > 0
\end{cases}$$

(3)

$$\sigma_{n_y} = \begin{cases} 
1 \sqrt{2} \cos \left( \frac{n_y \pi y}{b} \right) \sqrt{1 - \frac{n_y^2}{b^2}} : & n_y = 0 \\
\sqrt{2} \cos \left( \frac{n_y \pi x}{b} \right) : & n_y > 0
\end{cases}$$

(4)

The corresponding eigenvalues are

$$\alpha_n = \sqrt{\left( \frac{n_x \pi}{a} \right)^2 + \left( \frac{n_y \pi}{b} \right)^2}$$

(5)

In the $z$-direction, the pressure can be expressed as

$$P_n(z) = A_n e^{-ikbx} + B_n e^{ikbx}$$

(6)

where

$$k_n = \begin{cases} 
-\sqrt{k^2 - \alpha_n^2} : & k^2 < \alpha_n^2 \\
\sqrt{k^2 - \alpha_n^2} : & k > \alpha_n
\end{cases}$$

(7)

is the wavenumber of the $n$th mode in the axial direction and $k$ is the free space wavenumber. We can see that the axial wavenumber will for certain values of $\alpha_n^2$ be imaginary, and the propagation in $z$ direction will be evanescent (exponentially damped). The sign of the root is chosen so that this is the case when $k_n$ is inserted into (6). The frequency where $k_n$ becomes real is called the cut-off (or sometimes cut-on) frequency of the corresponding mode, $k_c = \alpha_n$.

Derivations and more details about how the modes are coupled between each duct section, and information on how to implement the method in practice, can be found in [5, 6, 11].

### 2.1. Radiation Impedance

The radiation impedance at the mouth end of the horn needs to be known in order to fix the boundary conditions at this end of the horn [5]. In the multimodal case, the radiation impedance will be a matrix that describe the coupling from all velocity modes to all pressure modes at the horn–interface. The elements $Z_{mn}$ of this matrix are found as [6]

$$Z_{mn} = \frac{j \omega \rho}{2 \pi S^2} \int_S \int_S \frac{e^{-jkh}}{h} \times \psi_m(x_0, y_0) \psi_n(x, y) dS_0 dS$$

(8)

where $h = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. The originally quadruple integral of (8) can be simplified in various ways, depending on the mode functions and coordinate system used. See Zorumski [12] for a circular duct, and Kemp [6, 13] for rectangular ducts with symmetric and asymmetric modes. The symmetric mode case has been used in the calculations in this paper.

In order to save computation time, the radiation impedance for a fairly large range of discrete $ka$ values was precomputed, and scaling and interpolation used thereafter.

### 2.2. Radiated Pressure

By exciting the horn throat with a given volume velocity distribution in modal form, and then propagating this volume velocity through the horn, the mouth volume velocity mode amplitudes $U_m$ can be found. From this, the radiated pressure can be calculated using a multimodal variant of the Rayleigh integral:

$$p(x,y,z) = \frac{j \omega \rho}{2 \pi S} \int 0 \int S U_m \psi_m(x_0, y_0) \frac{e^{-jkh}}{h} dS_0$$

(9)

### 2.3. Edge Diffraction

In order to simulate horns with small baffles or flanges, the influence of the pressure diffracted from the edges on the radiation impedance and radiated pressure, must be taken into account. A method for calculating first order diffraction impulse response based on the Biot-Tolstoi expressions has been presented by Svensson et al. [14], and the method has also been given a frequency domain formulation [15], and extended to include higher order diffraction [16]. The methods described in these papers are implemented in a freely available Matlab toolbox [17]. A modified beta version of this toolbox was used for the computation of edge diffraction effects in this paper.

The toolbox gives the transfer function from each point source to each receiver, and the pressure from a single point source is set as

$$p = Q \frac{e^{-jkr}}{r}$$

(10)

where $Q = \rho A/4\pi = 1$, and $A = j \omega U$ is the volume acceleration.

To compute the contribution of edge diffraction on the radiation impedance, we must perform the double surface integral of (8) over the horn mouth surface, with the exception that the term $e^{-jkh}/h$ is replaced by the edge diffraction transfer function from $(x_0, y_0)$ to $(x, y)$. A simplified geometry has been used, where the exterior of the horn is represented only by an infinitely thin plate corresponding to the flange. A grid of 24 by 24 points distributed according to the Gauss-Legendre quadrature rule was placed over the area of the horn mouth on one side of the plate. Each point was used both as source and as receiver.

When all transfer functions had been computed, the integral of (8) was performed using the Gauss-Legendre rule. This produced an impedance matrix giving the contribution of the edge diffraction, which was then added to the radiation impedance for a duct
In addition to the surface points, a set of receiver points in front of the horn was also included. The pressure at these points were calculated from (9) with the term $e^{-jkh}/h$ replaced by the edge diffraction transfer function, and the term $2\pi$ replaced by $4\pi$. Finally, this pressure was added to the pressure for the infinite baffle case calculated from (9). In both cases, the volume velocity amplitudes at the mouth were computed with the total radiation impedance taken into account.

3. Measurements

To verify the computational results, a square rectangular horn was built, the cross-sectional area following the hyperbolic-exponential horn profile of Salmon-type horns [18]. The horn used in the experiments has $S_t = 42.25 \text{ cm}^2$, $S_m = 1188.80 \text{ cm}^2$, $T = 0.7$ and a cutoff frequency of 200 Hz. The length of the horn is 0.5 m, and it is fitted with a 0.5 m flange that fits into a large baffle. Measurements were made of both throat impedance and frequency response at various points in front of the horn.

3.1. Setup

The setup is shown in Figure 1. The loudspeaker unit, a SEAS 11F-GX 4" midrange unit, is mounted in a small closed cabinet (hatched) filled with acoustic foam. This combination is connected to the horn through a 100 mm long impedance tube (black) in which two Brüel & Kjær 4149 microphones, M1 and M2, are mounted. The loudspeaker is driven by a signal from the WinMLS measurement system through a Lynx soundcard and a Quad 50E power amplifier.

The radiated pressure was measured with a Brüel & Kjær 4190 microphone.

The horn was either mounted in a 1255 mm by 1361 mm baffle, or free standing. The horn was offset from the center of the baffle, the center of the mouth was located at (735, 762) mm.

All measurements were done in an anechoic chamber.

3.2. Throat Impedance

The throat impedance is measured using the conventional two-microphone method, which is the standard method of measuring the acoustical impedance of absorbing material, mufflers and horns. The method, including the calibration, is described in detail in ISO 10534-2 [19].

It is known that the measurement results become unreliable above the first mode cutoff frequency, and at frequencies where $\Delta x = \lambda/2$. The first mode cuts in at about 2.6 kHz, and the microphones are one-half wavelength apart at 5.0 kHz.

3.3. Frequency Response

Frequency response was measured at several points in front of the horn, both on- and off-axis, with the impedance tube in place. Measurements were done in the near field, for several reasons. First, because the near field is harder to get right in the simulations, due to the fact that many of the evanescent modes that do not propagate to the far field are still present. Second, since the baffle is not infinite, and diffraction effects will be less prominent when measuring in the near field. Lastly, it was the most practical under the circumstances.

4. Comparisons

In this section, comparisons of measured and simulated values will be given. The measurements were compared to both the MMM, MMM with Edge Diffraction (MMM+ED), and two Boundary Element Methods (BEM). For the flanged horn, ordinary BEM was used [21], with a simple pyramid-shaped enclosure around the horn. For the baffled horn, the Boundary Element Rayleigh Integral Method (BERIM) [22] was used. BERIM combines the Boundary Element Method for the interior of the horn with a Rayleigh integral formulation for the exterior. This method is ideal for simulating horns mounted in an infinite baffle. An infinite baffle was also assumed in MMM simulation of the baffled horn, i.e. the large but finite baffle is treated as an infinite baffle in the simulations. The
results will show that this is an acceptable approximation.

The influence of edge diffraction was calculated for the frequency range 100Hz–2kHz. Extrapolation of the data is used below this range, but this did not work well for the higher frequencies, so above 2kHz the diffraction contribution is ignored.

It should be noted at this point that the focus so far has been the accuracy of the methods, and that computational efficiency has not been compared at this point. MMM and ED are implemented in Matlab, and the other two methods in two different compiled languages. A fair comparison would only be possible when all methods are optimized.

4.1. Throat Impedance

The throat impedance for the test horn mounted in a large baffle is shown in Figure 2. Measurements (solid lines) are compared to MMM (dashed) and BERIM or BEM (dotted).

\[ \text{Frequency [Hz]} \]

\[ Z_{th,n}(0,0) \] (a) Magnitude (normalized)

\[ \angle Z_{th,n}(0,0) \] (b) Phase

Figure 2: Throat impedance for the test horn in a large/infinte baffle

\[ Z_{th}, \text{measured} \]

\[ Z_{th}, \text{MMM} \]

\[ Z_{th}, \text{BERIM} \]

\[ \angle Z_{th}, \text{measured} \]

\[ \angle Z_{th}, \text{MMM} \]

\[ \angle Z_{th}, \text{BERIM} \]

\[ \angle Z_{th}, \text{measured} \]

The introduction of the first mode at approx. 2.6 kHz can be seen, and one notices the increased deviation from the simulated values above this frequency. The measurements break down completely above approximately 4.2 kHz, as opposed to the theoretical 5 kHz. This is, however, most likely due to the finite dimensions of the 0.5" microphones, since the 33.7 mm distance is the center-center spacing.

16 modes in each direction (a total of 256 modes) have been used in this simulation, and the horn consisted of 100 duct sections. For the BERIM simulation, a mesh bandwidth of 3kHz has been used, and the symmetry of the geometry is exploited.

The throat impedance for the test horn without any baffle, just a small flange, is shown in Figure 3. The same deviation above 2.6kHz can be seen. 14 modes in each direction have been used for the simulation, as the modal decomposition of the diffraction-related radiation impedance with the given number of integration points did not allow for modes of higher order to be reliably resolved. As in the infinite baffle case, the horn consisted of 100 duct sections. For the BEM simulation, a mesh bandwidth of 2kHz has been used, and the symmetry of the geometry is exploited.

The mesh bandwidth is understood as the frequency where the largest element of the mesh is not larger than 1/6th of a wavelength.
As expected, the impedance ripple is higher for the horn with flange than for the horn in the large baffle. This is mainly due to the reduced value of the radiation impedance at low frequencies for the flanged case. The MMM+ED seems to overestimate the first impedance peaks a little. The most likely reason for this is that the geometry of the horn is approximated by a single thin plate representing the flange, and the outside of the horn is ignored.

Figure 4: Radiated pressure for the test horn in a large/infinite baffle

Figure 5: Radiated pressure for the test horn with a small flange

4.2. Radiated Pressure

To make comparisons between simulations and measurements that are independent of the behavior of the loudspeaker driver used, a point at the center of the horn mouth has been used as reference. This point has been designated as the origin, and the pressures at all other points are compared to the pressure here. Figure 4 shows the relative responses at three different points in front of the baffled horn: one very close, one at a distance on the principal axis, and one that is out to the side and fairly close to the baffle. It
can be seen that the MMM and BERIM simulations follow each other well, and both capture the principal features of the measurements. The deviation at low frequencies is most likely due to the influence of the finite baffle used in the measurements. This is also most likely the cause of the deviation at 1kHz in Figure 4c; since this point is quite close to the baffle edge.

Figure 5 shows the response at the same points in front of the horn with small flange. Again, both simulations capture the principal features of the measurements. Above 2kHz the BEM simulation experiences problems with eigenfrequencies, so results for frequencies above 2kHz are not shown. For points in front of the horn, MMM holds up well to about 3kHz, but the deviation increases markedly above 2kHz for positions out to the side due to the lack of contribution from diffraction. Below 100Hz, the extrapolated edge diffraction contribution is most likely responsible for the deviation. The deviation of the MMM results in the 200–600Hz range in Figure 5c could be due to the approximate geometry used in the edge diffraction simulation, since at this point the rear of the horn is visible from the microphone position.

5. CONCLUSIONS

By expressing the sound field in a horn as a sum of eigenmodes, the throat impedance and radiated pressure from a loudspeaker horn can be computed with good accuracy. This paper has demonstrated that the method compares well with other numerical methods, like the Boundary Element Method and the Boundary Element Rayleigh Integral method, and that it captures the principal features of the radiated pressure. Throat impedance is also computed with good accuracy, enabling accurate computation of radiated power.

The addition of edge diffraction calculations has enabled the method to be used also for horns with small baffles or flanges, and the results compares well with the measurements.

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References